Université Paris Dauphine
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Introduction to Time series
TD4 - Spectral Analysis

Exercice 1 Suppose X and Y are two uncorrelated stationary process. Give the spectral measure of $X + Y$ as a function of those of X and Y.

Exercice 2 Determine the spectral measure of a MA(1) process, i.e. a stationary process X defined by $X_t = Z_t + \theta Z_{t-1}$ where $\theta \in \mathbb{R}$ and Z is a white noise.

Exercice 3

1. Determine the spectral measure of a AR(1) process, i.e. a stationary process X verifying $X_t = \phi X_{t-1} + Z_t$ where Z is a white noise with mean 0 and variance σ^2 and $\phi \notin \{-1, 1\}$.
2. Show that, if $\phi \neq \{-1, 1, 0\}$, there is a white noise $\tilde{Z}$ such that the stationary solution of $\tilde{X}_t = \phi^{-1}\tilde{X}_{t-1} + \tilde{Z}_t$ has the same autocovariance function as X. Give the variance of $\tilde{Z}$.

Exercice 4 Let $(f(h))_{h\in\mathbb{Z}}$ and $(g(h))_{h\in\mathbb{Z}}$ be two even and $\mathbb{R}$-positive definite real sequences. Prove that fg is also positive definite. We can proceed in two ways:

1. Build a stationary process for which fg is the autocovariance function.
2. Build a finite measure on $[-\pi, \pi[$ such that fg is the Fourier transform of this measure (we can assume that $(f(h))$ is in ℓ^1).

Let $\rho \in]0, 1[$, $\theta \in \mathbb{R}$ and $\gamma(h) = \rho^{|h|}\cos(\theta h)$ for all $h \in \mathbb{Z}$.

3. Show that $(\gamma(h))_{h\in\mathbb{Z}}$ is positive definite.
4. Specify its spectral measure and construct an explicit process having $(\gamma(h))_{h\in\mathbb{Z}}$ as function of autocovariance.

Exercice 5 Consider an ARMA equation

$$Q(B)X = P(B)Z$$

(B being the backward shift operator) with P and Q two polynomials without common roots, and Q without roots of modulus 1, so that the equation admits a unique stationary solution (with $Z \in BB(0, 1)$).

1. Under what condition on Q is the solution causal?
2. The solution X is said to be invertible when there exists a filter $(a_n) \in \ell^1$ such that

$$Z_t = \sum_{k=0}^{\infty} a_n X_{t-k} \qquad \forall t \in \mathbb{Z}.$$

   Under what condition on P is the solution invertible ? And under what additional condition is the filter (a_n) causal ?

3. We assume in this question that neither P nor Q have roots of modulus 1. Give an ARMA equation
$$\tilde{Q}(B)X = \tilde{P}(B)Z$$
whose solution has the same autocorrelation as that of the original equation, i.e. causal, invertible and inversely causal.

4. Apply this principle to the equation
$$X_t - 2X_{t-1} = Z_t + 3Z_{t-1}.$$